7.4 Solving Exponential Equations Using Logarithms

Keep in Mind

- If two expressions are equal, then the logarithms of those expressions are also equal: if \( M = N \), then \( \log_{10} M = \log_{10} N \), where \( M > 0, N > 0 \).
- You can use one of these three methods to solve an exponential equation:
  - If possible, write both sides of the equation with the same base, set the exponents equal to each other, and solve for the unknown.
  - Take the logarithm of each side and solve for the unknown.
  - Use graphing technology, using the systems of equations strategies you have employed for other kinds of equations.
- Some calculators will only calculate logarithms with base 10. Even so, you can evaluate any logarithm with base \( b \) using the change of base formula:
  \[
  \log_b x = \frac{\log x}{\log b}
  \]

Example 1

Solve the following exponential equation:

\[
4^{x + 2} = 31
\]

Round to three decimal places.

Solution

Step 1. Since 31 is difficult to write as a power of 4, I took the logarithm of both sides.

\[
4^{x + 2} = 31
\]

\[
\log 4^{x + 2} = \log 31
\]

Step 2. I rewrote the expression on the left using the Power Law of Logarithms.

\[
(x + 2) \log 4 = \log 31
\]

Step 3. I isolated the expression with \( x \).

\[
\frac{(x + 2) \log 4}{\log 4} = \frac{\log 31}{\log 4}
\]

\[
x + 2 = \frac{\log 31}{\log 4}
\]

\[
x = \frac{\log 31}{\log 4} - 2
\]

Step 4. I evaluated for \( x \), using a calculator.

To three decimal places, \( x = 0.477 \).
Example 2

Solve the following exponential equation. Round to two decimal places.

\[ 5^x + 2 = 7^x - 1 \]

Solution

Step 1. Since the two sides of the equation cannot be written with the same base, I took the logarithm of both sides.

\[ \log 5^x + 2 = \log 7^x - 1 \]

Step 2. I rewrote each side using the Power Law of Logarithms.

\[(x + 2) \log 5 = (x - 1) \log 7 \]

\[ x \log 5 + 2 \log 5 = x \log 7 - \log 7 \]

Step 3. I isolated the expressions with \( x \).

\[ 2 \log 5 + \log 7 = x \log 7 - x \log 5 \]

\[ 2 \log 5 + \log 7 = x (\log 7 - \log 5) \]

\[ \frac{2 \log 5 + \log 7}{\log 7 - \log 5} = x \]

Step 4. I evaluated for \( x \), using a calculator.

To two decimal places, \( x = 15.35 \).

Practice

1. Estimate the value of \( x \) in each equation to one decimal place. Then solve the equation. Show your work. Round your answer to three decimal places.

   a) \( 10 = 4^x \)

   \[ \log 10 = \log 4^x \]

   \[ \log 10 = x \log 4 \]

   \[ x = \frac{\log 10}{\log 4} = 1.661 \]

   b) \( 4.2^x = 20 \)

   \[ \log 4.2^x = \log 20 \]

   \[ x \log 4.2 = \log 20 \]

   \[ x = \frac{\log 20}{\log 4.2} = 2.087 \]

2. Estimate the value of \( x \) in each equation to one decimal place. Then solve the equation. Show your work. Round your answer to three decimal places.

   a) \( 40 = 5(4^x) \)

   \[ \frac{40}{5} = 4^x \]

   \[ \log 8 = \log 4^x \]

   \[ \log 8 = x \log 4 \]

   \[ x = \frac{\log 8}{\log 4} = 1.500 \]

   b) \( 60 = 100\left(\frac{1}{4}\right)^x \)

   \[ x = \frac{\log 0.6}{\log \left(\frac{1}{4}\right)} \]

   \[ 0.6 = \left(\frac{1}{4}\right)^x \]

   \[ \log 0.6 = x \log \left(\frac{1}{4}\right) \]

   \[ \log 0.6 = x \log \left(\frac{1}{4}\right) \]

   \[ x = 0.368 \]
3. Estimate the value of each logarithm, and then evaluate to three decimal places, using the change of base formula.

\[ \log_2 12 \] \hspace{1cm} \[ \log_8 2 \] \hspace{1cm} \[ \log_6 40 \]

\[ \frac{\log 12}{\log 2} = -3.858 \] \hspace{1cm} \[ \frac{\log 2}{\log 8} = 0.333 \] \hspace{1cm} \[ \frac{\log 40}{\log 6} = 2.059 \]

4. Write each expression as a base 10 logarithm. Evaluate to three decimal places.

\[ \log_4 40 \] \hspace{1cm} \[ \log_6 1000 \]

\[ \frac{\log 40}{\log 4} = 2.601 \] \hspace{1cm} \[ \frac{\log 1000}{\log 6} = -3.855 \]

\[ \log_2 \frac{3}{8} \]

\[ \frac{\log 3}{\log 8} = -1.415 \] \hspace{1cm} \[ \log_{0.2} 400 \]

\[ \frac{\log 400}{\log 0.2} = -3.723 \]

5. Solve each equation, and round your answer to two decimal places.

\[ 6^{x+1} = 22 \] \hspace{1cm} \[ \left( \frac{2}{3} \right)^{-x} = 12 \]

\[ \log 6^{x+1} = \log 22 \] \hspace{1cm} \[ \log \left( \frac{2}{3} \right)^{-x} = \log 12 \]

\[ (x+1) \log 6 = \log 22 \] \hspace{1cm} \[ -x \log \left( \frac{2}{3} \right) = \log 12 \]

\[ x+1 = \frac{\log 22}{\log 6} \] \hspace{1cm} \[ -x = \frac{\log 12}{\log (\frac{2}{3})} \]

\[ x+1 = 1.725 \] \hspace{1cm} \[ -x = -6.129 \]

\[ x = 0.725 \] \hspace{1cm} \[ x = 6.129 \]

6. Freya has $3400 in an investment that earns 5% interest, compounded annually. Determine the number of years it will take for her balance to surpass $5000. Use the compound interest formula \( A = P(1 + i)^n \), where \( A \) represents the future value, \( P \) represents the present value, \( i \) represents the interest rate per compounding period, and \( n \) represents the number of compounding periods. Show your calculations.

\[ \frac{5000}{3400} = \left( 1 + 0.05 \right)^n \]

\[ 1.470588235 = 1.05^n \]

\[ \log 1.470588235 = \log 1.05^n \]

\[ \log 1.470588235 = n \log 1.05 \]

\[ n = \frac{\log 1.470588235}{\log 1.05} \]

\[ n \approx 7.9 \text{ yrs} \]
7. Solve each equation, and round your answer to two decimal places.

a) \(6^x - 1 = 3^{x+1}\)
\[
\log_6(6^x - 1) = \log_6(3^{x+1})
\]
\[
(x-1)\log_6 = (x+1)\log_3
\]
\[
x\log_6 - \log_6 = x\log_3 + \log_3
\]
\[
x\log_6 - x\log_3 = \log_3 + \log_6
\]
\[
x(\log_6 - \log_3) = \log_3 + \log_6
\]
\[
x = 4.17
\]

b) \(10^{x-2} = 7^{x+1}\)
\[
\log_{10}(10^{x-2}) = \log_{10}(7^{x+1})
\]
\[
(x-2)\log_{10} = (x+1)\log_7
\]
\[
x = 2\log_{10}7 + \log_{10}7 + 2
\]
\[
x = 18.37
\]

8. The healing of a wound with an initial area of 60 cm² can be modelled by the function \(A(t) = 60(10^{-0.023t})\), where \(A(t)\) represents the area of the wound, in square centimetres, after \(t\) days of healing. In how many days will 50% of the wound be healed? Show your calculations.

\[
30 = 60(10^{-0.023t})
\]
\[
0.5 = 10^{-0.023t}
\]
\[
\log 0.5 = \log 10^{-0.023t}
\]
\[
\log 0.5 = -0.023t \cdot \log 10
\]
\[
0.301 = -0.023t
\]
\[
t = 13.09 \text{ days}
\]

9. Which is closest to the value of \(x\) in the following exponential equation?

\[
5^{x-1} = 4^{x+2}
\]

A. 0.05  
B. 2.1  
C. 4.2  
D. 19.6

10. Which is closest to the value of \(x\) in the following exponential equation?

\[
7^{x-2} = 3^{x+2}
\]

A. 7.2  
B. 2.7  
C. 2.9  
D. 1.0

11. Kim has invested $7000 at 7.6% interest, compounded quarterly. The investment will be worth at least $10 000 after \(19\) quarters, or \(4.75\) years and \(57\) months.

12. $5000 is invested at 3.5% interest, compounded annually. The investment will have doubled in value after \(20\) years.

13. Jolene currently has $3900 in credit card debt. The interest rate on her credit card is 19.5%, compounded daily. If she makes no payments against the balance, her debt will have doubled after \(1298\) days, or about \(3.56\) years.
Notes

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# 11.

\[ A = P \left(1 + \frac{r}{n}\right)^{nt} \]

\[ \frac{10000}{7000} = \left(1 + \frac{0.076}{4}\right)^x \]

\[ 1.429 = (1 + 0.019)^x \]

\[ 1.429 = 1.019^x \]

\[ \log 1.429 = \log 1.019^x \]

\[ \log 1.429 = x \log 1.019 \]

\[ x = \frac{\log 1.429}{\log 1.019} = 18.95 \text{ compounds} \]

\[ \therefore 19 \text{ compounds} \]

\[ \therefore \text{19 quarters} \]

\[ \frac{19}{4} = 4.75 \text{ yrs} \]

\[ 4.75 \times 12 = 57 \text{ months} \]

# 12.

\[ \frac{1000}{5000} = \frac{5000(1 + 0.035)^x}{5000} \]

\[ 2 = 1.035^x \]

\[ \log 2 = \log 1.035^x \]

\[ \log 2 = x \log 1.035 \]

\[ x = \frac{\log 2}{\log 1.035} \]

\[ x = 20.15 \text{ yrs} \]

# 13.

\[ 7800 = 3900 \left(1 + \frac{0.195}{365}\right)^x \]

\[ 2 = (\frac{1.000534}{1.000534})^x \]

\[ \log 2 = x \log \left(\frac{1.000534}{1.000534}\right) \]

\[ \frac{\log 2}{\log 1.000534} = x \]

\[ 1298 \text{ days} = x \]

\[ \frac{1298}{365} = 3.56 \text{ yrs} \]