2.4 Permutations When Objects Are Identical

Today's Focus: Determine the number of permutations when some objects are identical

INVESTIGATION

Consider the words BIKE and BOOK. How does the number of ways that the four letters in each word can be arranged differ? Why?

BIKE, IBEK, KBIE, EIBK
BIEK, IBEK, KBIE, EIBK
BEIK, IEKB, KEIB, EKIB
BEKI, IEKB, KEBI, EKBI

BOO, OOBK
BKO, OOBK
BKOO, OOBK
KOBO, OOBK

Book are not distinguishable so Book is the same as Boo, etc.

12 ways

4! = 24

KEY IDEAS!!!

There are fewer permutations when some of the objects in a set are identical compared to when all the objects in a set are different. This is because some of the arrangements are identical.

The number of permutations of $n$ objects, where $a$ are identical, another $b$ are identical, another $c$ are identical, and so on, is

$$P = \frac{n!}{a!b!c!}$$

For example, in the set of four objects $a, a, b,$ and $b$, the number of different permutations, $P$, is

$$P = \frac{4!}{(2! \cdot 2!)} = 6$$

The six different arrangements are $aabb, bbaa, abab, baba, abba,$ and $baab$. 

“Principles of Mathematics 12” Nelson Education
THINGS TO REMEMBER!!!

Dividing \( n! \) by \( a!, b!, c! \), and so on deals with the effect of repetition caused by objects in the set that are identical. It eliminates arrangements that are the same and that would otherwise be counted multiple times.

Example 1: Evaluate the following expressions

a) \( \frac{7!}{3!2!} = 420 \) arrangements

b) \( \frac{8!}{2!2!2!} = 5040 \) arrangements

Example 2: In the mountainous regions of India, China, Nepal, and Bhutan, it is common to see prayer flags. Each flag has a prayer written on it, and colour is used to symbolize different elements: green (water), yellow (earth), white (air/wind), blue (sky/space), and red (fire). How many different arrangements of the same prayer can you make using these 9 flags: 1 green, 1 yellow, 2 white, 3 blue, and 2 red?

\[ \frac{9!}{2!3!2!} = 15120 \] arrangements
Example 3: How many ways can the letters of the word CANADA be arranged, if the first letter must be N and the last letter must be C?

\[
\frac{1 \times 4 \times 3 \times 2 \times 1 \times 1}{3!} = \frac{24}{6} = 4\ \text{ways}
\]

A repeat A's

Example 4: Julie's home is three blocks north and five blocks west of her school. How many routes can Julie take from home to school if she always travels either south or east?

\[
\text{8!} \leftarrow \text{total # of blocks}
\]

\[
\frac{5! \times 3!}{\text{repeats}}
\]

56 ways